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MODULATION AND PARSEVAL'S RELATION OF GENERALIZED FRACTIONAL SINE TRANSFORM

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Abstract: Transforms with cosine and sine functions as the transform kernels represent an important area of analysis. It is based on the so-called half-range expansion of a function over a set of cosine or sine basis functions. Because the cosine and the sine kernels lack the nice properties of an exponential kernel, many of the transform properties are less elegant and more involved than the corresponding ones for the Fourier transform kernel. As the sine transform, cosine transform and Hartley transform are widely use in signal processing, the application of their fractional version in signal/image processing is very promising. This paper concerned with generalized one dimensional fractional Sine transforms and here we discuss Modulation theorem, Parseval's identity for generalized one dimensional fractional Sine transform.

Keywords: *Fractional Fourier Transform, Fractional Cosine Transform, Fractional Sine Transform.*

I.INTRODUCTION:

The mathematical transforms such as Fourier transform, wavelet transform and Fractional Fourier transform have long been influential mathematical tools in information processing. These transforms process signal from time to frequency domain or in joint time–frequency domain. The Discrete Fractional transforms have been comprehensively and systematically treated from the signal processing point of view. The significant features of Discrete Fractional transform benefit from their extra degree of freedom that is provided by Fractional orders. Comparison of performance states that Discrete Fractional Fourier transform is superior in compression, while Fractional Cosine transform is better in encryption of image and video. Mean square error and peak signal-to-noise ratio with optimum Fractional order are considered quality check parameters in image and video.

We know, the Modulation theorem of integral transform is a linear phase shift in time domain results in a frequency domain and the Parseval's theorem is the integral of the square of a function is equal to the integral of the square of its transform. Parseval's theorem to solve some definite integrals. In fact, the applications of this theorem are extensive and can be used to easily solve many difficult problems. Modulation is the process of merging two signals to form a third signal with desirable characteristics of both. This always involves nonlinear processes such as multiplication; you can't just add the two signals together. In radio communication, modulation results in radio signals that can propagate long distances and carry along audio or other information.

We know that the Cosine and Sine transforms and their Discrete versions are useful tools in signal and image processing, such as signal coding [4], watermarking [5] and restoration of de-focused images [6]. In the Ref. [7] Pei and Yeh extended the Cosine transform to the Discrete Fractional Cosine transform (DFrCT) and the Discrete Fractional Sine transform (DFrST). Moreover, the DFrCT and DFrST are used in the digital computation of FrFT for the reducing computational load of theDFrFT. The success of FrFT in its application has promoted the development of other kinds of Fractional transforms like fractional Hartley transform, fractional Hadamard transform and Fractional Cosine transform, Fractional Sine transform (FrST). Pei Soo-Chang redefined the Fractional Cosine transform and Fractional Sine transform based on Fractional Fourier transform in 2001 [8-9]. FrST is the extension of Sine transform and it has been widely used in domain of digital signal and image processing.

II.MATHEMATICAL PRE-REQUISITES

2.1. One dimensional generalized Fractional Sine transform

One dimensional Fractional Sine transform with parameter $\alpha f(x)$ denoted by $FC_\alpha(u)$ perform a linear operation given by the integral transform.

$$FS_\alpha\{f(x)\}(u)=\int_{-\infty}^{\infty} f(x)\, k_\alpha(x,u)\,dx$$

(2.1)

Where the kernel,

$$k_\alpha(x,u)=\sqrt{\frac{1-icot\alpha}{2\pi}}\,e^{\frac{i\left(x^2+u^2\right)cot\alpha}{2}}\,e^{i\left(\alpha-\frac{\pi}{2}\right)}\sin\left(cosec\alpha.ux\right)$$

(2.2)

2.2. The test function space E

An infinitely differentiable complex valued function $\emptyset$ on R^n belongs to (R^n) . If for each compact set $I\subset S_a$ where,

$$S_a=\{x_i:x\in R^n,|t|\le a,a>0\},\ I\in R^n$$

$$\gamma_{E_p}(\emptyset)=\sup_{x\in I}\left|D_x^p\emptyset(t)\right|<\infty,\ \text{where, p =1, 2, 3….}$$

Thus $E(R^n)$ will denote the space of all $\emptyset\in E(R^n)$ with support contained in S_a

Note that: The space E is complete and therefore a Freechet space. Moreover, we say that f is a Fractional Sine transformable, if it is a member of E^*, the dual space of E.

2.3 Distributional one-dimensional Fractional Sine transform

The one dimensional distributional Fractional Sine transform of $f(x) \in E^*(R^n)$ defined by

$$FS_\alpha\{f(x)\} = \langle f(x), k_\alpha(x,u)\rangle \qquad (2.4)$$

$$k_\alpha(x,u) = \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{i(x^2+u^2)cot\alpha}{2}}\,\mathrm{e}^{\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}\sin(cosec\alpha.ux)\,. \qquad (2.5)$$

Where , RHS of equation (2.4) has a meaning as the application of $f \in E^*$to$K_\alpha(t,u,) \in E$.

III.COME-PROPERTIES

3.1 Parseval's Identity

If $FS_\alpha\{f(x)\}(u) = FS_\alpha(u)$and $FS_\alpha\{g(x)\}(u) = GS_\alpha(u)$

Then$i$) $\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx = 4\,\mathrm{e}^{-2\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}\int_{-\infty}^{\infty}\overline{GS_\alpha(u)}\,FS_\alpha\{f(x)\}(u)du$

ii)$\int_{-\infty}^{\infty}|f(x,y)|^2\,dx = \int_{-\infty}^{\infty}|FS_\alpha(u,v)|^2\,4\,\mathrm{e}^{-2\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}du$

Solution: --By definition of generalized one dimensional fractional Sine transform

$$FS_\alpha\{g(x)\}(u) = GS_\alpha(u) = \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{i(u^2)cot\alpha}{2}}\mathrm{e}^{\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}\int_{-\infty}^{\infty} g(x)\,e^{\frac{i(x^2)cot\alpha}{2}}\sin(cosec\alpha.ux)\,.dx$$

Using inversion formula of one dimensional fractional Sine transform

$$g(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} GS_\alpha(u)\,e^{-\frac{i(x^2+u^2)cot\alpha}{2}}\mathrm{e}^{-\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}\sqrt{\frac{2\pi}{1-icot\alpha}}s\,in(cosec\alpha.ux)\,.cosec\alpha du$$

$$= \frac{2}{\pi}\sqrt{\frac{2\pi}{1-icot\alpha}}e^{-\frac{i(x^2)cot\alpha}{2}}\mathrm{e}^{-\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}cosec\alpha\int_{-\infty}^{\infty} GS_\alpha(u)\,e^{-\frac{i(u^2)cot\alpha}{2}}s\,in(cosec\alpha.ux)\,.du$$

$$\therefore\ \overline{g(x)} = \frac{2}{\pi}\sqrt{\frac{2\pi}{1+icot\alpha}}e^{\frac{i(x^2)cot\alpha}{2}}\mathrm{e}^{-\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}cosec\alpha\int_{-\infty}^{\infty}\overline{GS_\alpha(u)}\,e^{\frac{i(u^2)cot\alpha}{2}}s\,in(cosec\alpha.ux)\,.du$$

$$\therefore\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx = \int_{-\infty}^{\infty} f(x)dx\frac{2}{\pi}\sqrt{\frac{2\pi}{1+icot\alpha}}e^{\frac{i(x^2)cot\alpha}{2}}\mathrm{e}^{-\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}cosec\alpha\int_{-\infty}^{\infty}\overline{GS_\alpha(u)}\,e^{\frac{i(u^2)cot\alpha}{2}}s\,in(cosec\alpha.ux)\,.du$$

$$= \sqrt{\frac{8}{\pi(1+icot\alpha)}}cosec\alpha\,\mathrm{e}^{-\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{\frac{i(x^2+u^2)cot\alpha}{2}}\sin(cosec\alpha.ux)\,.\overline{GS_\alpha(u)}dudx$$

$$= \sqrt{\frac{16}{(1+cot^2\alpha)}}cosec\alpha\,\mathrm{e}^{-2\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}\int_{-\infty}^{\infty}\overline{GS_\alpha(u)}du$$

$$\int_{-\infty}^{\infty}\sqrt{\frac{1-icot\alpha}{2\pi}}f(x)e^{\frac{i(x^2+u^2)cot\alpha}{2}}\,\mathrm{e}^{\mathrm{i}\left(\alpha-\frac{\pi}{2}\right)}s\,in(cosec\alpha.ux)\,.dx$$

$$= \sqrt{\frac{16}{(1+cot^2\alpha)}}cosec\alpha\, e^{-2i\left(\alpha-\frac{\pi}{2}\right)} \int_{-\infty}^{\infty} \overline{GS_\alpha(u)}\, FS_\alpha\{f(x)\}(u)du$$

$$= \frac{4cosec\alpha}{csc\alpha}\, e^{-2i\left(\alpha-\frac{\pi}{2}\right)} \int_{-\infty}^{\infty} \overline{GS_\alpha(u)}\, FS_\alpha\{f(x)\}(u)du$$

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx = 4\, e^{-2i\left(\alpha-\frac{\pi}{2}\right)} \int_{-\infty}^{\infty} \overline{GC_\alpha(u)}\, FS_\alpha\{f(x)\}(u)du$$

ii) Let $f(x) = g(x)$ then

$FS_\alpha(u) = GS_\alpha(u)\overline{FS_\alpha(u)} = \overline{GS_\alpha(u)}$

$$\int_{-\infty}^{\infty} f(x)\,\overline{f(x)}dx = 4\, e^{-2i\left(\alpha-\frac{\pi}{2}\right)} \int_{-\infty}^{\infty} FS_\alpha(u)\,\overline{FS_\alpha(u)}du$$

$$\int_{-\infty}^{\infty} |f(x,y)|^2\, dx = \int_{-\infty}^{\infty} |FS_\alpha(u,v)|^2\, 4\, e^{-2i\left(\alpha-\frac{\pi}{2}\right)}du$$

3.2 Modulation Property - I

$$\boldsymbol{FS_\alpha\{f(x)\,cosax\}(u)}$$
$$\boldsymbol{= e^{\frac{i}{2}(u^2)cot\alpha}\sqrt{\frac{1-icot\alpha}{1-icot\theta}}\, e^{i(\alpha-\theta)}\{e^{\frac{-iP^2cot\theta}{2}}FS_\alpha\left\{f(x)e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}\right\}(P)}$$
$$\boldsymbol{- e^{\frac{-iR^2cot\theta}{2}}FS_\alpha\left\{f(x)e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}\right\}(R)\}}$$

Proof: consider,

$$FS_\alpha\{f(x)cosax.\}(u) = \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{i(u^2)cot\alpha}{2}}\, e^{i\left(\alpha-\frac{\pi}{2}\right)} \int_{-\infty}^{\infty} f(x)cosax\, e^{\frac{ix^2cot\alpha}{2}}\, sin(cosec\alpha.ux)\, dx$$

$$FS_\alpha\{f(x)cosax.\}(u) = AB\int_{-\infty}^{\infty} f(x)cosax\, e^{\frac{ix^2cot\alpha}{2}}\sin(cosec\alpha.ux)\, dx$$

where $A = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{i\left(\alpha-\frac{\pi}{2}\right)}$, B$= e^{\frac{i(u^2)cot\alpha}{2}}$

$$FS_\alpha\{f(x)cosax.\}(u) = AB\int_{-\infty}^{\infty} f(x)\, e^{\frac{ix^2cot\alpha}{2}}\frac{(e^{icsc\alpha ux}+e^{-icsc\alpha ux})}{2}\frac{(e^{iax}-e^{-iax})}{2i}dx$$

$$= \frac{AB}{4i}\int_{-\infty}^{\infty} e^{\frac{ix^2cot\alpha}{2}}\begin{pmatrix}(e^{i(csc\alpha u+a)x}-e^{-i(csc\alpha u+a)x})\\ -(e^{i(csc\alpha u-a)x}-e^{-i(csc\alpha u-a)x})\end{pmatrix} f(x)dx$$

$$= \frac{AB}{2i}\{ \int_{-\infty}^{\infty} e^{\frac{ix^2cot\alpha}{2}} 2i\sin((csc\alpha u+a)x\, f(x)dx - \int_{-\infty}^{\infty} e^{\frac{ix^2cot\alpha}{2}} 2isin(csc\alpha u-a)x\, f(x)dx\}$$

Let $(csc\alpha u+a) = csc\theta.P$, $(csc\alpha u-a) = csc\theta.R$,

$$= AB\{ \int_{-\infty}^{\infty} e^{\frac{ix^2cot\alpha}{2}} \sin(csc\theta.Px)\, f(x)dx - \int_{-\infty}^{\infty} e^{\frac{ix^2cot\alpha}{2}} \sin(csc\theta.Rx)\, f(x)dx\}$$

$$= e^{\frac{i}{2}(u^2)cot\alpha}\sqrt{\frac{1-icot\alpha}{1-icot\theta}}\, e^{i\left(\alpha-\frac{\pi}{2}\right)}\, e^{-i\left(\theta-\frac{\pi}{2}\right)}$$
$$\left\{\int_{-\infty}^{\infty} e^{\frac{ix^2(cot\alpha-cot\theta)}{2}} e^{\frac{i(x^2+P^2)cot\theta}{2}} e^{\frac{-iP^2cot\theta}{2}}\, e^{i\left(\theta-\frac{\pi}{2}\right)} sin(csc\theta.Px)\right) f(x)dx$$
$$- \int_{-\infty}^{\infty} e^{\frac{ix^2(cot\alpha-cot\theta)}{2}} e^{\frac{i(x^2+R^2)cot\theta}{2}} e^{\frac{-iR^2cot\theta}{2}}\, e^{i\left(\theta-\frac{\pi}{2}\right)} sin(csc\theta.Rx)\, f(x)dx\}$$

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$$FS_\alpha\{f(x)\,cosax\}(u)$$
$$= e^{\frac{i}{2}(u^2)cot\alpha}\sqrt{\frac{1-icot\alpha}{1-icot\theta}}\,e^{i(\alpha-\theta)}\{e^{\frac{-iP^2cot\theta}{2}}FS_\alpha\left\{f(x)e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}\right\}(P) - e^{\frac{-iR^2cot\theta}{2}}FS_\alpha\left\{f(x)e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}\right\}(R)\}$$

3.3 Modulation Property- II

$$\boldsymbol{FS_\alpha\{f(x)sinax\}(u)}$$
$$\boldsymbol{=-\frac{e^{\frac{i}{2}(u^2)cot\alpha}}{2}\sqrt{\frac{1-icot\alpha}{1-icot\theta}}e^{i\left(\theta-\frac{\pi}{2}\right)}\{\int_{-\infty}^{\infty}e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}e^{\frac{i(x^2+P^2)cot\theta}{2}}e^{\frac{-iP^2cot\theta}{2}}\cos(csc\theta\,.\,Px)f(x)dx - \int_{-\infty}^{\infty}e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}e^{\frac{i(x^2+R^2)cot\theta}{2}}e^{\frac{-iR^2cot\theta}{2}}\cos(csc\theta\,.\,Rx)\,f(x)dx\}}$$

Proof: Consider

$$FS_\alpha\{f(x)sinax\}(u) = \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{i(u^2)cot\alpha}{2}}\,e^{i\left(\alpha-\frac{\pi}{2}\right)}\int_{-\infty}^{\infty}f(x)sinax.\,e^{\frac{i(x^2)cot\alpha}{2}}\sin(coseca.\,ux)\,.\,dx$$

$$= AB\int_{-\infty}^{\infty}f(x)sinax\;e^{\frac{ix^2cot\alpha}{2}}\sin(coseca.\,ux)\,dx$$

Where $A = \sqrt{\frac{1-icot\alpha}{2\pi}}\,e^{i\left(\alpha-\frac{\pi}{2}\right)}$, B= $e^{\frac{i(u^2)cot\alpha}{2}}$

$$= AB\int_{-\infty}^{\infty}f(x)\,e^{\frac{ix^2cot\alpha}{2}}\frac{(e^{icsc\alpha ux}-e^{-icsc\alpha ux})}{2i}\frac{(e^{iax}-e^{-iax})}{2i}dx$$

$$=-\frac{AB}{4}\int_{-\infty}^{\infty}e^{\frac{ix^2cot\alpha}{2}}\begin{pmatrix}(e^{i(csc\alpha u+a)x}+e^{-i(csc\alpha u+a)x})\\-(e^{i(csc\alpha u-a)x}+e^{-i(csc\alpha u-a)x})\end{pmatrix}f(x)dx$$

$$=-\frac{AB}{4}\int_{-\infty}^{\infty}e^{\frac{ix^2cot\alpha}{2}}(2cos((csc\alpha u+a)x-2cos(csc\alpha u-a)x)f(x)dx$$

Let $(csc\alpha u+a) = csc\theta.P$, $(csc\alpha u-a) = csc\theta.R$

$$=-\frac{AB}{2}\left\{\int_{-\infty}^{\infty}e^{\frac{ix^2cot\alpha}{2}}\cos(csc\theta\,.\,Px)\right)f(x)dx - \int_{-\infty}^{\infty}e^{\frac{ix^2cot\alpha}{2}}\cos(csc\theta\,.\,Rx)\;f(x)dx\}$$

$$=-\frac{e^{\frac{i}{2}(u^2)cot\alpha}}{2}\sqrt{\frac{1-icot\alpha}{1-icot\theta}}e^{i\left(\theta-\frac{\pi}{2}\right)}\{\int_{-\infty}^{\infty}e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}e^{\frac{i(x^2+P^2)cot\theta}{2}}e^{\frac{-iP^2cot\theta}{2}}\cos(csc\theta\,.\,Px)f(x)dx - \int_{-\infty}^{\infty}e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}e^{\frac{i(x^2+R^2)cot\theta}{2}}e^{\frac{-iR^2cot\theta}{2}}\cos(csc\theta\,.\,Rx)\,f(x)dx\}$$

$$FS_\alpha\{f(x)sinax\}(u)$$
$$=-\frac{e^{\frac{i}{2}(u^2)cot\alpha}}{2}\sqrt{\frac{1-icot\alpha}{1-icot\theta}}e^{i\left(\theta-\frac{\pi}{2}\right)}\{\int_{-\infty}^{\infty}e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}e^{\frac{i(x^2+P^2)cot\theta}{2}}e^{\frac{-iP^2cot\theta}{2}}\cos(csc\theta\,.\,Px)f(x)dx - \int_{-\infty}^{\infty}e^{\frac{ix^2(cot\alpha-cot\theta)}{2}}e^{\frac{i(x^2+R^2)cot\theta}{2}}e^{\frac{-iR^2cot\theta}{2}}\cos(csc\theta\,.\,Rx)\,f(x)dx\}$$

Application: Using integral transform the work also has been carried out by Steven W. Smith (9) and gives the proper modulation of signals. **Modulation** is the process of merging two signals to form a third signal with desirable characteristics of both. This always involves nonlinear processes such as multiplication; you can't just add the two signals together. In radio communication, modulation results in radio signals that can propagate long distances and carry along audio or other information. Radio communication is an extremely well developed discipline, and many modulation schemes have been developed. One of the simplest is called amplitude **modulation**.

IV.Conclusion:

In the present work we have proved Parsevel's identity and Modulation Property for generalized fractional Sine transform.

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