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# Probabilistic Entropy Measure Derived by using Quadratic Polynomials and their properties

**Surender Kumar[1], Omdutt Sharma[2], Naveen Kumar[3]**
[1]*Research Scholar, Department of Mathematics, Baba Mast Nath University, Rohtak 124001, India*
[2]*Assistant Professor, Department of Mathematics, P.D.M. University, Bahadurgarh 124507, India.*
[3]*Professor, Department of Mathematics, Baba Mastnath University, Asthal Bohor Rohtak 124021, India*
[1]*surenderkumar00481@gmail.com*, [2]*omdutt86@gmail.com*, [3]*naveenkapilrtk@gmail.com*

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**Abstract:** First, C. E. Shannon introduced Shannon's entropy, an entropy measure in communication theory. This measure is logarithmic in nature. In order to quantify information uncertainty, various academics developed new logarithmic and exponential entropy metrics after Shannon. In this study, a novel probabilistic entropy measure that efficiently measures complexity and uncertainty in complex systems is proposed using the quadratic equation. These novel probabilistic entropy Metrics have a big impact on how we understand complicated systems and how we make decisions in many fields. Several established entropy axioms have been used to verify the validity of the new probabilistic entropy measure. The findings show that quadratic entropy metrics perform better than current ones in capturing minute variations in system uncertainty and behavior. In this paper, we discuss some properties of this measure.

**Keywords:** Entropy, fuzzy set, uncertainty measure, quadratic function, information measure.

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## I.INTRODUCTION:

To quantify the degree of uncertainty in information, Shannon [1948] developed the first information metric, called entropy information. In situations where the random variable's value is unknown, the average information content is measured using this logarithmic probabilistic entropy. This Shannon Entropy, or information entropy, is defined as $H_S(P) = -\sum_{i=1}^{n} p_i \log p_i$, The entropy measure is continuous for probability and is never negative. It reaches its highest value, logn, when all possible events are equally likely. The primary disadvantage of this metric is that Shannon's entropy may be negative for certain probability distributions. It no longer qualifies as an uncertainty measure because of this restriction. Because of the importance of entropy in information theory, entropy literature developed quickly.

A parametric generalization of Shannon's entropy was presented by Renyi [1961] after the original. Subsequently, researchers developed other parametric generalizations, such as Havrda and Charvat [1967] entropy, Kapur [1967,1997] entropy, and Tsallis [1988] entropy. These are all logarithmic entropies. As $p_i \to 0, then -\log p_i \to \infty$. Although the event information is always finite, this is the situation of infinite; hence, further assumptions are needed. In keeping with this perspective, Pal & Pal [1989, 1992, 1999, 1991(a)] suggested an additional entropy measure in exponential form. It is defined as follows: $H_{PP}(P) = \sum_{i=1}^{n} p_i e^{(1-p_i)}$. It is always non-negative and for $p_i = 1/n$ It always has a definite upper bound e and is independent of n. Afterward, a generalized exponential entropy of order-α was proposed by Kvalseth [2000]. Application of Entropy to a Lifetime Model was introduced by Awad [1987]. Using a conic-section equation, Sharma [2019] introduced probabilistic entropy metrics and their use in dimension reduction. The Use of the Entropy Function for Validating Results in Differential Calculus was introduced by Sharma [2024]. Different entropy metrics, such as trigonometric and hyperbolic measures, have been proposed by scholars in the literature. An additional benefit of these entropy measures is their applicability to a variety of issues, including pattern recognition, picture extraction, feature selection, feature evaluation, image thresholding, and more. In order to overcome the drawbacks of conventional entropy measures, this manuscript suggests a novel probabilistic entropy measure based on a quadratic function. With potential uses in a number of domains, such as machine learning, image analysis, and signal processing, the suggested probabilistic entropy measure is intended to offer a more sensitive and reliable assessment of uncertainty.

**Theoretical Ground:**

This section provides a concise overview of entropy, which is a measure of a random variable's uncertainty. Let the probability mass function of a discrete random variable X with support S be $p(x) = P(X = x), x \in S$, Notable definitions include the following:

Shannon proposed the first logarithmic entropy, which has the following definition:

**The definition of a discrete random variable X's initial logarithmic entropy is as follows:**

$$H_{L1}(X) = -\sum_{x \in S} p(x) \log p(x) \qquad (2.1)$$

This entropy is expressed in bits with a log base of 2.

Let $f_X(x)$ be the density function of the continuous random variable X in the continuous case. The entropy of this density function is as follows:

$$h_{l1}(X) = -\int_S f_X(x) \log f_X(x)\, dx \qquad (2.2)$$

We refer to this entropy as differential entropy. The primary flaw with this entropy measure is that it is not always non-negative; in fact, it can occasionally be negative, in which case it ceases to be an uncertainty measure.

**Another logarithmic entropy in literature is as follows:**

Renyi [1961] introduces the entropy specified in (2.3).

$$H_{L2}(X) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} p_i^{\alpha}, \alpha > 0, \alpha \neq 1 \qquad (2.3)$$

Kapur [1967] introduces the entropy specified in (2.4).

$$H_{L3}(X) = \frac{1}{\beta-\alpha} \ln \frac{\sum_{i=1}^{n} p_i^{\alpha}}{\sum_{i=1}^{n} p_i^{\beta}}, \alpha \neq \beta. \qquad (2.4)$$

Kapur [1986] introduces the entropy specified in (2.5).

$$H_{L4}(X) = -\sum_{i=1}^{n} p_i \ln p_i + \frac{1}{a}(1+ap_i)\ln(1+ap_i) - \frac{1}{a}(1+a)\ln(1+a), \ a \geq -1 \qquad (2.5)$$

Pal and Pal [1989] created the exponential entropy measure with respect to Shannon entropy, which is defined as follows, to get around the drawbacks of the logarithmic entropy measure:

**Definition: For a discrete random variable X, the first exponential entropy is defined as:**

$$H_{E1}(X) = \sum_{x \in S} p(x) e^{(1-p(x))} \qquad (2.6)$$

Let $f_X(x)$ be the density function of the continuous random variable X in the continuous case. The entropy of this density function is as follows:

$$h_{e1}(X) = \int_S f_X(x) e^{(1-f_X(x))} dx \qquad (2.7)$$

The advantage of this exponential entropy is that it is always non-negative.

Using the conic section equation, Sharma and Gupta [2019] developed some new entropy metrics.

$$H_{Sc}(P) = \frac{1}{n}\sum_{i=1}^{n} \sqrt{(0.5)^2 - (p_i - 0.5)^2}; \qquad -(2.8)$$

$$H_{e1}(P) = \frac{1}{n}\sum_{i=1}^{n} b * \sqrt{1 - \frac{(p_i - 0.5)^2}{(0.5)^2}}; where\ 0 < b < 0.5; \quad -(2.9.A)$$

$$H_{e2}(P) = \frac{1}{n}\sum_{i=1}^{n} a * \sqrt{1 - \frac{(p_i - 0.5)^2}{(0.5)^2}}; where\ 0.5 < a; \quad -(2.9.B)$$

$$H_P(P) = -\frac{1}{4an}\sum_{i=1}^{n} [(p_i - 0.5)^2 - 0.25]; where\ a > 0 \qquad -(2.10)$$

$$H_{Hb}(P) = -\frac{a}{bn}\sum_{i=1}^{n} \left[\left(\sqrt{b^2 + (p_i - 0.5)^2}\right) - \frac{1}{2}\left(\sqrt{4b^2 + 1}\right)\right]; where\ a, b > 0 \ (2.11)$$

Here $p_i \in P\ and\ 0 \leq p_i \leq 1$.

These axioms are satisfied by a good probabilistic entropy measure:

1. **Non-negativity:** Entropy is non-negative in its domain.
2. **Minimality:** Entropy is minimum (i.e., 0) in certain situations ($p_i = 0\ or\ 1$.
3. **Maximality:** Entropy is maximum in most uncertain situations ($p_i = 0.5$).
4. **Resolution:** Entropy is increasing at $p_i \in [0, 0.5]$ and decreasing at $p_i \in [0.5, 1]$.
5. **Symmetric:** $H(p) = H(1-p)$.
6. **Continuity:** Entropy is a continuous function of the probability distribution $P$.
7. **Symmetry:** Symmetric for permutations of $p_1, p_2, p_3, \ldots.. p_n$.
8. **Maximality:** The entropy measure will be maximal if all the outcomes are equally likely, i.e.
   i. $H(p_1, p_2, p_3, \ldots.. p_n) \leq H\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots . \frac{1}{n}\right)$.

## II.PROPOSED WORK:

In this section, a quadratic polynomial is used to construct a novel probabilistic entropy measure. The definition of a generic quadratic function is: $f(x) = ax^2 + bx + c$ (3.1)

where $a \neq 0$, and $a, b, c$ are constant.

If in equation (3.1), $c = 0$ *and* $a = -1$ *and* $b = 1$ then equation (3.1) is changed as

$$f(x) = -x^2 + x = (x - x^2) \qquad (3.2)$$

The proposed probabilistic entropy measure for any probability distribution 'P' is now defined as follows using equation (3.2):

$$H_Q(P) = \sum_{i=1}^{n} (p_i - {p_i}^2) \qquad (3.3)$$

Here, $P$ is the probability distribution, and $p_i \in P$ & $0 \leq p_i \leq 1$ *and* $\sum p_i = 1$

A valid probabilistic entropy measure meets all of the axioms listed in Section 2 and is specified in (3.3). It is stated as a theorem that the suggested probabilistic entropy measure is legitimate.

**Theorem 1:** Prove that the probabilistic entropy measure $H_Q(P)$ defined in (3.3) is a valid probabilistic entropy measure.

**Proof:** This theorem can be proved by using the following lemmas:

**Lemma 1:** $H_Q(P)$ is non-negative, i.e., $H_Q(P) = \sum_{i=1}^{n}(p_i - {p_i}^2) \geq 0$.

**Proof:**

$$H_Q(P) \geq 0 \; iff \; \sum_{i=1}^{n} (p_i - {p_i}^2) \geq 0 \; iff \; (p_i - {p_i}^2) \geq 0$$

$$iff \;\; p_i(1 - p_i) \geq 0 \; iff \;\; p_i \geq 0 \;\; and \; (1 - p_i) \geq 0 \; that \; is \;\; p_i \leq 1.$$

It is true because of how probability is defined. $H_Q(P)$ is hence non-negative.

**Lemma 2:** $H_Q(P) = \sum_{i=1}^{n}(p_i - {p_i}^2) = 0 \; i.e. minimum \; iff \; p_i = 0 \; or \; 1$

**Proof:**

$$H_Q(P) = \sum_{i=1}^{n} (p_i - {p_i}^2) = 0 \; iff \;\; (p_i - {p_i}^2) = 0$$

$$iff \;\; p_i(1 - p_i) = 0 \;\; iff \; p_i = 0 \; or \; 1.$$

Hence $H_Q(P)$ is the minimum for certain cases.

**Lemma 3:** $H_Q(P) = \sum_{i=1}^{n}(p_i - {p_i}^2)$ is maximum in most uncertain situations, i.e. $p_i = 0.5$.

**Proof: The** second-order derivative method is used to check that $H_Q(P)$ is maximum at $p_i = 0.5$. Differentiate $H_Q(P)$ concerning $p_i$.

$$\frac{d(H_Q(P))}{d(p_i)} = 1 - 2p_i = 0;$$

$$1 - 2p_i = \mathbf{0} \Rightarrow p_i = 0.5 \; .$$

$$\frac{\mathbf{d^2}(H_Q(P))}{\mathbf{d}{p_i}^{\mathbf{2}}} = \mathbf{-2 < 0}$$

Hence $H_Q(P)$ is maximum at $p_i = 0.5$.

**Lemma 4:** $H_Q(P)$ is increasing at $p_i \in [0, 0.5]$ and decreasing at $p_i \in [0.5, 1]$ $i.e.$ $H_Q(P)$ satisfies the resolution property.

**Proof:** It can be proved by the application of the derivative

$$\frac{\mathbf{d}H_Q(P)}{\mathbf{d}p_i} = \sum_{i=1}^{n} [\mathbf{1} - 2p_i]$$

$$\frac{\mathbf{d}H_Q(P)}{\mathbf{d}p_i} \geq \mathbf{0} \; at \; p_i \in [0, 0.5] and \; \frac{\mathbf{d}H_Q(P)}{\mathbf{d}p_i} \leq \mathbf{0} \text{ at } p_i \in [0.5, 1]$$

Accordingly, it is claimed that $H_Q(P)$ is rising at $p_i \in [0, 0.5]$ and falling at $p_i \in [0.5, 1]$.

**Lemma 5:** $H_Q(p) = H_Q(1 - p)$ that is measure is symmetric (Dual) in nature.

**Proof:**

$$H_Q(1 - p_i) = \sum_{i=1}^{n} ((1 - p_i) - (1 - p_i)^2)$$

$$= \sum_{i=1}^{n} (1 - p_i - 1 - {p_i}^2 + 2p_i) = \sum_{i=1}^{n} (p_i - {p_i}^2) = H_Q(P)$$

Hence $H_Q(P)$ is symmetric (Dual) in nature.

**Lemma 6:** $H_Q(p)$ is a continuous function of $P$.

**Proof:** Since $H_Q(p)$ It is an algebraic function, and an algebraic function is continuous. So $H_Q(p)$ is a continuous function of $P$.

**Lemma 7:** $H_Q(p)$ will be maximal if all the outcomes are equally likely $\left(p_i = \frac{1}{n}\right)$ i.e.

$$H_Q(p_1, p_2, p_3, \ldots..p_n) \leq H_Q\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots.\frac{1}{n}\right).$$

**Proof:** For $n$ In the equally probable event, the entropy is

$$H_Q(P) = \sum_{i=1}^{n}(p_i - p_i{}^2) = \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{1}{n^2}\right) = 1 - \frac{1}{n}$$

Because of this, the probabilistic entropy measure is dependent on the number of equiprobable occurrences; the more equiprobable events there are, the higher the entropy value.

**Lemma 8:** $H_Q(P)$ is symmetric to the permutations of $p_1, p_2, p_3, \ldots..p_n$, i.e.

$$H_Q(p_1, p_2, p_3, \ldots..p_n) = H_Q(p_3, p_1, p_2, \ldots..p_n)$$

**Proof:**

$$H_Q(p_1, p_2, p_3, \ldots..p_n) = \sum_{i=1}^{n}(p_i - p_i{}^2) = \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} p_i{}^2$$

$$= (p_1 + p_2 + p_3 + \cdots..+p_n) - (p_1{}^2 + p_2{}^2 + p_3{}^2 + \cdots..+p_n{}^2)$$

$$(p_3 + p_1 + p_2 + \cdots..+p_n) - (p_3{}^2 + p_1{}^2 + p_2{}^2 + \cdots..+p_n{}^2) = H_Q(p_3, p_1, p_2, \ldots..p_n)$$

Thus $H_Q(P)$ is symmetric for permutations of $p_1, p_2, p_3, \ldots..p_n$.

Because every lemma listed above demonstrates the assumptions of a valid entropy measure. Thus, a valid entropy measure is $H_Q(P)$.

## III.PROPERTIES:

Sharma et al. [2024] proposed some extra properties as a validation of entropy measures:

**9.** Entropy measure verified the rolle's theorem.

**10.** The entropy measure is neither increasing nor decreasing on (0,1).

**11.** Entropy measure verified the Iagrange's mean-value theorem.

The proposed entropy also satisfied the condition proposed by Sharama et al. [2024], shown below:

**Lemma 9:** Prove that the entropy measure $H_Q(P)$ verified Rolle's theorem.

**Proof:** For verification of Rolle's Theorem $H_Q(p)$ The entropy measure satisfied the following condition:

1. $H_Q(P)$ is continuous on [0, 1];
2. $H_Q(P)$ is differentiable on (0, 1); and
3. $H_Q(0) = H_Q(1)$;

then, there must exist at least one point $p \in (0, 1)$ such that $H_Q{}'(p) = 0$

Since $H_Q(p)$ is an algebraic polynomial, and an algebraic polynomial is continuous everywhere, and hence $H_Q(p)$ is a continuous in $[0, 1]$.

We know that $H_Q(P) = \sum_{i=1}^{n}(p_i - p_i{}^2)$, Differentiating for $p_i$ , $\frac{d(H_Q(P))}{d(p_i)} = 1 - 2p_i$, which exists for all $p_i \in$ (0,1). Thus $H_Q(p)$ is derivable in (0,1).

$H_Q(0)$ =0 and $H_Q(1)$ =0, this implies $H_Q(0) = H_Q(1)$ .

Since all the conditions are satisfied, there must exist at least one point $p \in$ (0, 1) such that $H_Q{}'(p) = 0$.

$$\frac{d(H_Q(P))}{d(p_i)} = 1 - 2p_i$$

Now $H_Q'(p) = 0$ which implies $1 - 2p = 0$ which implies $p = 0.5 \in$ (0,1).

Hence, Rolle's theorem is verified.

**Lemma 10:** The entropy measure $H_Q(P)$ is neither increasing nor decreasing on (0,1).

**Proof:** Here $H_Q(P) = \sum_{i=1}^{n}(p_i - p_i{}^2)$

Differentiating for $p_i$ , $H_Q'(p) = 1 - 2p_i$.

The entropy measure $H_Q(P)$ will be increased if $H_Q'(p)$ >0 on (0,1).

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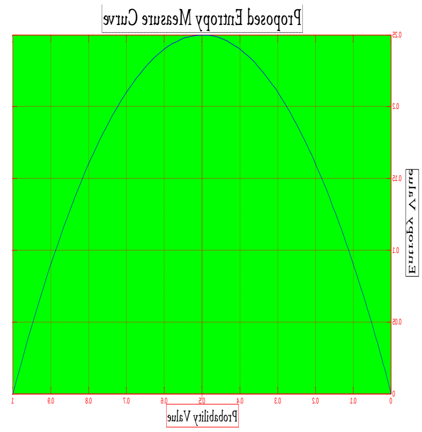
That is, if $1 - 2p_i > 0$ i.e., $1 > 2p_i$ i.e.,$p_i < \frac{1}{2}$.

The entropy measure $H_Q(P)$ will be decreased if $H_Q'(p) < 0$ on (0,1).

That is, if $1 - 2p_i < 0$ i.e., $1 < 2p_i$ i.e.,$p_i > \frac{1}{2}$.

This shows that the entropy measure $H_Q(P)$ is increasing on (0, 0.5 ) and decreasing on ( 0.5 ,1 ) and hence, the entropy measure $H_Q(P)$ is neither increasing nor decreasing on (0,1).

*Figure1:*




The suggested entropy measure curve is increasing on (0,0.5) and decreasing on (0.5,1), as seen in Fig. 1. Consequently, on (0,1), the entropy measure $H_Q(P)$ is neither rising nor falling.

**Lemma 11:** Entropy measure verified the Iagrange's mean-value theorem.

**Proof:** For verification of Lagrange's mean-value theorem $H_Q(p)$ The entropy measure satisfies the following condition:

1. $H_Q(P)$ is continuous on [0, 1];
2. $H_Q(P)$ is differentiable on (0, 1);

Then, there must exist at least one point $p \in (0,1)$ such that $H_Q'(p) = \frac{H_Q(1)-H_Q(0)}{1-0}$.

Since $H_Q(p)$ being a polynomial is continuous for all real $p$ and hence $H_Q(p)$ is a continuous in $[0,1]$.

Here $H_Q(P) = \sum_{i=1}^{n}(p_i - {p_i}^2)$, Differentiating for $p_i$ , $H_Q'(p) = 1 - 2p_i$, which exists for all $p_i \in$ (0,1). Thus $H_Q(p)$ is derivable in (0,1).

Since all the conditions are satisfied, there must exist at least one point $p \in$ (0, 1) such that $H_Q'(p) = \frac{H_Q(1)-H_Q(0)}{1-0}$.

Now, $H_Q'(p) = 1 - 2p$ and $H_Q(1) = 0$ and $H_Q(0) = 0$.

This implies, $1 - 2p = \frac{H_Q(1)-H_Q(0)}{1-0} = 0$, which implies $p = 0.5 \in (0,1)$.

Hence, Iagrange's mean-value theorem is verified.

## IV.CONCLUSION:

This research paper presents a novel probabilistic entropy measure based on a quadratic polynomial, addressing the limitations of existing entropy measures. The proposed measure demonstrates desirable properties, including non-negativity and sensitivity to changes in probability distributions. Its effectiveness is validated through theoretical analysis and potential applications in various fields. This contribution advances the field of information theory, offering new avenues for uncertainty quantification and decision-making under uncertainty. Future research directions include exploring the measure's applications in real-world scenarios and further refining its properties.

**Declaration of Conflicting Interests:**

The authors declare no conflicts of interest

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