|| Volume 6 || Issue 7 || July 2021 || ISSN (Online) 2456-0774
INTERNATIONAL JOURNAL OF ADVANCE SCIENTIFIC RESEARCH
AND ENGINEERING TRENDS

REVIEW OF FRACTIONAL INEQUALITIES WITH APPLICATIONS OF FRACTIONAL CALCULUS

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Abstract: Fractional Calculus (FC) goes back to the inception of the theory of differential calculus. Nevertheless, the application of FC just developed in the recent two decades, due to the advancement in the domain of chaos that revealed subtle links with the FC notions. At the topic of dynamical systems theory, some work has been carried out but the proposed models and methods are still in a preliminary state of establishment. Having these notions in mind, the paper explores FC in the study of system dynamics and control. In this context, this work analyses the usage of FC in the fields of controller tuning, legged robots, redundant robots, heat diffusion, and digital circuit synthesis. We explored review of fractional inequalities.

Keywords: *Fractional Calculus, Robot, Digital circuit synthesis, Engineering.*

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I INTRODUCTION

In modelling engineering and scientific challenges, fractional differential and integration balances play increasingly essential roles. In many circumstances, such models have been shown to produce more appropriate results than analogue models with integer derivatives. A thorough investigation has been made into the computation of the fractional-order derivatives and fractional differential equations [1]. The results for fractional differential equations are derived by a fixed-point technique in most of the accessible publications. The qualitative features of Riemann–Liouville (R–L) and the Caputo derivatives can be discovered using the differential and integral inequalities.

The generalisation in non-integer values of the concept of derivative $D^\alpha[f(x)]$ goes back to the early stage of differential calculus Theory [2]. Indeed, Leibniz had various comments regarding calculating the $D^{\frac{1}{2}}[f(x)]$ in his communication with Bernoulli, L'Hôpital and Wallis (1695). Nonetheless, numerous mathematicians, like Euler, Liouville, Riemann, and Letnikov, have contributed to the development of the theory of fractional calculus (FC).

The FC is concerned with arbitrary derivatives and components (real or, even, complex order) [6]. There have been various distinct approaches to the mathematical definition of a derivative/integral fractional order. For instance, the definition of Laplace of a fractional $x(t)$ signal derivative is

$$D^\alpha x(t) = L^{-1}\{s^\alpha X(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1}x(t)|t = 0\} \qquad \text{-------------(1)}$$

Where $n-1 < \alpha \leq n, \alpha > 0$. The Grünwald-Letnikov definition

$$D^\alpha x(t) = \lim_{h\to 0}\Big[\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} x(t-kh)\Big] \qquad \text{-------------(2)}$$

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$$

Where $\Gamma(x)$ is the function of the gamma and h is the increase of the time. However, (2) reveals that "global operators" have a memory of past occurrences and are suitable for memory effects modelling in most materials and systems.

The definition of a fractional-order derivative for Riemann-Liouville is $(\alpha > 0)$:

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(c)}{(t-c)^{\alpha-n+1}}dr,\ n-1 < \alpha < n \qquad \text{--------------(3)}$$

is given by ($\alpha \in \mathfrak{R}$):
Where $\Gamma(x)$ is the Gamma function of x.

The fractional-order integrals/derivatives of numerous functions can be calculated on the basis of the provided definitions. Nevertheless, it is not straightforward to determine and implement fractional-order algorithms, and the next parts will address the issue.
FC has been a successful field of scientific and engineering research in recent years. Many scientific sectors are actually taking the concept of FC into account and we may use it in viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos, fractals, heat transfer, biology, electronics, signal processing, robotics, system identification, traffic systems, genetic algorithms, percolation, modelling and identification, telecommunications. Telecommunication systems are also included [9].

Taking these notions into account, FC's various science and engineering applications. The application of FC idea for tuning PID controllers is described, and the fractional dynamics of the trajectory control of redundant manipulators are provided in the application of a fractional-order PD controller for controlling the leg joints of the hexapod robot [7]. Next, the fractional properties of heat diffusion are introduced along with a media, and the application of FC to circuit creation with evolutionary algorithms is displayed.

Preliminaries:

1. M. Caputo and M. Fabrizio proposed $\alpha \in (0,1)$ of a totally integrative function f on $[t_0, \tilde{T}]$, for any $\tilde{T} > t_0 \geq 0, f \in AC_{loc}([t_0, \infty)$ (which means the first of the original derivatives of f is integral on $[0, \tilde{T})$ for any $\tilde{T} > 0$, in the form

$$ {}^{CF}_{t_0}D^{\alpha}_{t} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_{t_0}^{t} \exp\left(-\frac{\alpha}{(1-\alpha)}(t-r)\right) f^{(1)}(r)\,dr, \quad t > t_0, $$

Where $M(\alpha)$ a normalizer function, such as $M(0) = M(1) = 1$ and $f^1 = \frac{d}{dr}f$

2. Almeida et al., considered the initial value problem of a fractional differential equation including ψ-Caputo fractional derivative

$$ {}^{c}D_{a^+}^{\alpha,\psi}\, y(t) = f(t, y(t)), t \in [a,b], $$

$$ y(a) = y_a, y_{\psi}^{[k]}(a) = y_a^k, k = 1,2,\dots,n-1, $$

Where $f: [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^{c}D_{a^+}^{\alpha,\psi}$ is the ψ-Caputo fractional derivative of order $n-1 < \alpha < n, y_a, y_a^k \in \mathbb{R}$ and $t \in [a,b], y \in C^{n-1}[a,b]$.

3. Anastassiou presents improvements of a composition rule for the Canvati fractional derivatives, which have the general form

$$ \int_a^b w_1(t)|(D^{\mu_1}f)(t)||(D^{\mu_2}f)(t)|dt \leq K\left(\int_a^b w_2(t)\,|(D^{u}f)(t)|^{p}\,dt\right)^{\frac{2}{p}}, $$

Where w_2 and w_1 are weight functions, and $D^{\gamma}f$ denotes the Canavati functional derivatives of f of order γ.

4. In Marchaud's doctoral thesis, fractional differentiation for sufficiently regular real functions $f: (0,1) \to \mathbb{R}$ extended with 0 for $x \geq 0$, whenever $\alpha \in (0,1)$:

$$ D^{\alpha}f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{(f(x)-f(x-t))}{t^{(1+\alpha)}}\,dt. $$

REVIEW OF LITERATURE:

Many mathematical experts such as Hardy et al. Beckenbach & Bellman discussed the origin of the inequalities in many works. Mitrinos . A detailed treatment of integral inequalities may be found in a work by B.G. Pachpatte that covers multiple applications in the theory of different types of differential and integrative equations. Gronwall showed an outstanding inequality that has gained great notice in the literature and still attracts interest. The Gronwalls type inequalities of a variable play a very important role in the qualitative theory of differential for real functions. R. Bellman is responsible for the first application of the Gronwall inequality in order to establish limitations and uniqueness. Gronwall-Bellman inequality is an essential tool for studying qualitative behaviour of solutions of differential and stability, which is normally shown in elementary differential equations using continuity arguments. Fractional inequality development is based on fractional calculus which originated in the L-interest Hospital's to know 1 of the derivatives x and Leibnitz 2 studied the answer in 1695. The fractional theory of Euler (1770) who formulated continuous differentiation by interpolation. The exponent law for fractional derivatives was developed by Lagrange's (1849). The factional derivative of an integral is defined by Laplace (1820) [8]. The derivative of arbitrary order was found first by Lacroix (1819), subsequently by Joseph B.J. Fourier[1822]. Niels Henrick Abel first uses fractional operation and then follows other definitions. The fractional analysis is a very useful means of differentiating and integrating with the real or complex numerical powers of differential or integrative operators in mathematical analysis. The research monographs by Miller & Ross (1993) Kiryakova (1994), Rubin (1975), Podlubny (1999) and Kilbas etal contain a complete accounting of fractional calculus operators together with their properties and application (2006) [1]. Mathai and Haubold (2008) and Mathai et al. were chapters alluded to by the author (2010). The monographs of R.Bellman(1965),V.Laxmikantham are used to start the development of integral inequality. A.A.Martynyuk &R.Gutowski (1979)J.Scroder (1980) etc. Leela(1969),W.Walter(1970),P.R.Be esack(1975), A.N.Filatov and L.V.SHAROV (1976). Our theory of integral inequalities is not sufficient to examine partial differential equations and integral differential equations. Thus fractional inequalities are developed through the functions of convex, concave, Mittag Leffler, Lyopunov, etc. The authors identified a few slightly unique integral inequalities of the type Grownwall-Bellman and applied them to qualitative analysis of the solutions to several fractional differential equations of the type Caputo. Lyapunov inequality and many of their generalisations have been proven valuable instruments in the theory of oscillations, disconnectedness, problems of self-value and many other applications in differential and differential equation theories as well as time scales [2]. One may note the third group of inequalities that are focused on the concrete functional analysis of individual components. Maximum function inequalities between Hardy and Littlewood were demonstrated in 1930. The

contemporary theory and study of partial differential equations is mostly based on the right spaces for the analysis and they are

related to the Sobolev theory of embedding and inequalities. For an insight into these inequities see Adams. The authorestablished inequalities of type Bihari with fractional derivatives. He revealed how to use this disparity to demonstratethe limits and the global lives of individual families and to identify asymptotic behaviour. In we identified a number of non-linear differential inequalities with fractional derivatives and suggested application for them. Belarbi, S,Dahmani,Z have explored in and created a certain number of novel fractional integral inequalities via the general operators of Liouville on fractional calculus. Many articles investigate the fractional integrals of the Riemann-Liouville region and present intriguing generalisations of the Hermite-Hadamard type. In the author J. Park created concepts like Hermite-Hadamard for n time distinctive functions which in the second sense were m-convex and s-convex [5]. W.J. Liu, F. Qi, B.Y. Xi produced inequalitieslike Simpson in recent years employing improved convex functions. A generalised integrative operator with the widespread Mittag-Leffler function has several fundamental inequalities: generalisations with various results shown. Many writers are currently researching inequalities between Riemann- Liouville, Caputo, Hilfer, Canvati and fractional integraloperators. In reality, uniqueness of the solution of the differentialequation for a fractional order can be found using fractional integral inequalities and upper and lower boundaries for solutions to the fractional boundary value problem.

Tuning of PID Controllers Using Fractional Calculus Concepts:

The most popular control algorithms in industry are PID controllers. The Ziegler-Nichols (Z-N) approach is the most popular and is still widely used for determining PID parameters among the different systems for tuning PID controllers. It is well known that compensated systems that use this approach with controllers usually have a step response with a significant percentage excess. In addition, only plants with monotonous step response are suited by the Z-N heuristics.

We examine in this section a PID tuning process in order that the reaction of the compensated system has a predefined value that is practically constantly overflowing. This approach is based on a minimum square error integral (ISE) between the unit feedback control system's step replies, the open-loop transfer $L(s)$ function provided by a fractional order integrator, and that of a compensated PID system.

Figure 1 shown the fractional-order control system utilised in the tuning of PID controllers as a reference model. The function open-loop transfer $L(s)$ is set to ($\alpha \in \mathbb{R}^{+}$):

$$L(s) = \left(\frac{\omega_c}{s}\right)^{\alpha} \qquad \text{-- - - - - - -- - - - - - - - -} (4)$$

Where the frequency $|L(j\omega_c)| = 1$ is the gain crossover. The α parameter is the curve slope on a log scale and can accept both integer and non-integer values. We consider in this research $1 < \alpha < 2$ to allow for a fractional oscillation in the output response (similar to an underdamped second-order system). This function of transfer is also known as the ideal loop transfer function of the Bode since in the 1940s Bode studied the construction of feedback amplifiers.

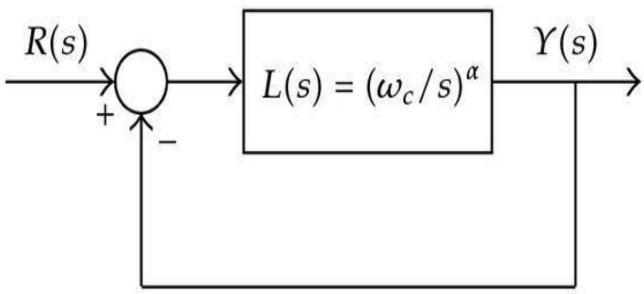

Figure 1: Fractional-order control system with open-loop transfer function $L(s)$.

Figure 2 provides an illustration of the amplitude and phase graphs of $L(s)$. The amplitude curve is a direct line with a constant pitch -20αdB/dec, with a phase curve being a horizontal line at $-\alpha\pi/2$. The Nyquist curve, $\arg L(j\omega) = -\alpha\pi/2$ Rad, is simply the line across the origin.

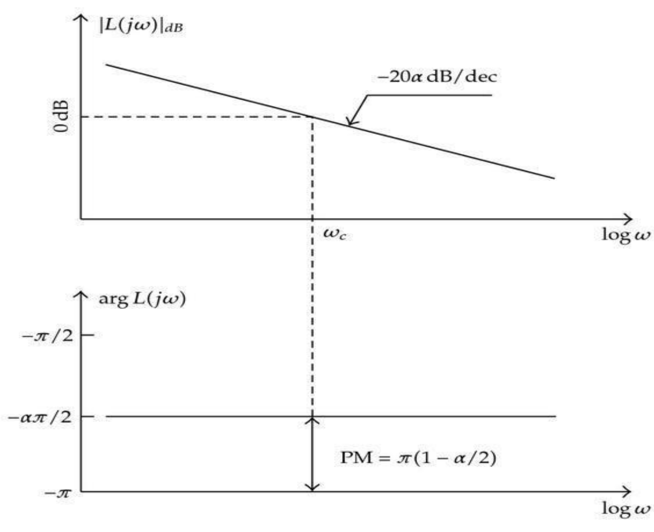

Figure 2: Bode diagrams of amplitude and phase of $L(j\omega)$ for 1<α<2.

This $L(s)$ choice provides a closed loop system that is insensitive to changes in its desired property. When the gain varies, the crossover frequency ω_c will change, but the $PM = \pi(1 - \alpha/2)$ rad is the phase margin of the system irrespective of the gain value. This is shown in the amplitude and phase curves of Figure 2.

Figure 1 gives the closed loop transfer function of the fractional control system

$$G(s) = \frac{L(s)}{1+L(s)} = \frac{1}{(s/\omega_c)^\alpha+1}, 1 < \alpha < 2 \quad \text{-------------- (5)}$$

The unit step response of $G(s)$ is given by the expression:

$$y_d(t) = L^{-1}\{\frac{1}{s}G(s)\} = L^{-1}\{\frac{\omega_c^\alpha}{s(s^\alpha+\omega_c^\alpha)}\} = 1 - \sum_{n=0}^{\infty}\frac{[-(\omega_c t)^\alpha]^n}{\Gamma(1+\alpha n)} = 1 - E_\alpha[-(\omega_c t)^\alpha] \quad \text{------ (6)}$$

We use the fractional-order transfer function (5) as the reference system in the tuning of PID controllers [3]. We can determine the excess shoot and the speed of the output response by order α and crossover frequency ω_c Correspondingly. To this end, we take into account the closed-loop system presented in Figure 3 where the PID and plant transfer functions are $G_c(s)$ and the $G_p(s)$.

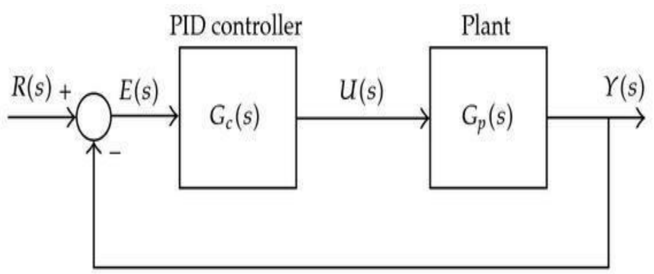

Figure 3: Closed-loop control system with PID controller $G_C(s)$.

We see that the step reactions exceed the frequency of the crossover practically constantly, independently of the fluctuation in the plant gain. Therefore, the proposed methodology can generate resilient closed-loop systems that can change and take steps to show an iso-damping feature. Several systems were evaluated to demonstrate good results for the proposed strategy. It also was compared with other tuning methods that showed similar or higher results.

Fractional P Control of a Hexapod Robot:

Walking machines permit the locomotion of another type of cars on terrain, which does not require a constant support platform, but at the expense of higher leg coordination and control needs. Joint level control is normally carried out via a PID-like system with position feedback for these robots. Applying FC's theory to robotics has recently revealed potential areas for future improvements. In this section, different fractional PD^α are compared Tuning of robot controllers for the walking system joint control (Figure 4) with n = 6 legs, distributed equally on each side and having three rotating joints each. The robot controllers were tuned [4].

The joint j=3 can be either mechanically driven or motorised during this research leg (Figure 4). We assume that there is a rotating pre-tensioned spring-dashpot connecting leg links L_{i2} and L_{i3} in the case of mechanically activated cases. This mechanical impedance keeps the angle between both bands when the joint torque is imposed.

The hexapod body and foot-ground contact dynamic model is presented in Figure 5. Robot body conformance is recognised because animals walking have a spine that enhances stability and promotes locomotion. The robot body is divided into n identical segments, and the intrabody compliance process is carried out by a linear spring-damper system (specified by parameters that resemble that of an animal). The ground contact of the robot is modelled using a non-linear system, which is based on soil mechanics studies for parameters.

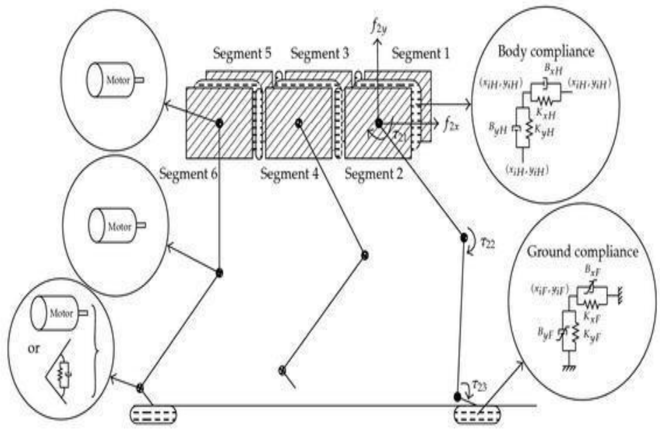

Figure 4: Model of the robot body and foot-ground interaction.

The general architecture of control is shown on the hexapod robot. In this work, we examine the impact of various implementations of PD^α, $\alpha \in \Re$ .Total controllers for $G_{c1}(s)$, whereas G_{c2} is a proportional control with $Kp_j = 0.(j = 1,2,3)$.

Again, we have a 4th-order approximation of the PD^{α} method performed in a discrete time

$$G_{c1j}(z) \approx Kp_j + K\alpha_j \frac{\sum_{i=0}^{i=u} a_{ij}\, z^{-i}}{\sum_{i=0}^{i=u} b_{ij}\, z^{-i}}$$

Where Kp_j and $K\alpha_i$ are the proportional and derivative gains, respectively, and α_j is the fractional-order, for joint j . Therefore, the classical PD^1 algorithm occurs when the fractional order $\alpha_j = 1.0$.

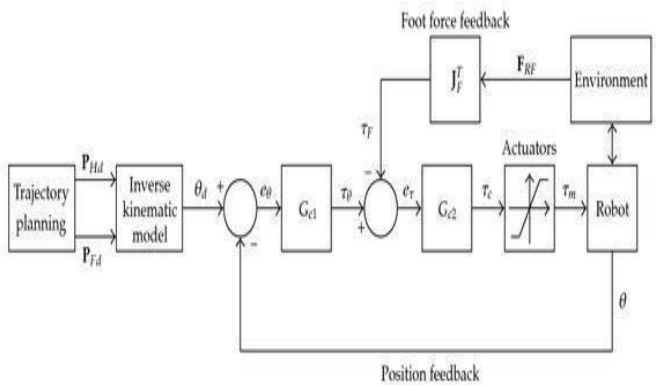

Figure 5: Hexapod robot control architecture

Application of fractional calculus

Some application of fictional calculus are given below-

1. Fractional conservation of mass
2. Fractional Control of a Hexapod Robot
3. Fractional Dynamics in the Trajectory Control of Redundant Manipulators
4. Heat diffusion
5. Groundwater flow of water

Application in Groundwater flow of water:

Used in hydrogeology, the mathematical relationship of groundwater flow across an aquifer is the groundwater flow. A version of diffusion equations similar to that used in heat shipment to define heat flow through a solid is provided for the transient flow of groundwater (heat conduction). A type of Laplace equation, which is a possible flow form and has comparable data in several domains, represents the steady-state flow of soil.

The groundwater flow equation is generally computed for a small representative elementary volume (REV), in which the medium parameters are supposed to be consistent effectively. The flux conditions in the relation are described ahead by applying the constituent equation termed the Law of Darcy, which demands that the flux be laminary. Other methods include agent-based models that have the effect of complicated aquifers like karstic or fractured rocks.

Conclusions and Future Scope of Study:

Fractional calculus (FC) goes back to the start of differential calculus theory. However, in the recent two decades, FC's application has surfaced due to the progress made in chaos, revealing subtle links to FC principles.

FC has recently been a successful scientific and engineering research topic, and many scientific fields are paying more attention to the FC concepts now. Some work has been carried out in dynamic systems theory, although the proposed models and algorithms continue to be established at an early stage. This article offered a number of case studies on FC-based models and control systems that demonstrate the advantages of employing FC theory in various research and engineering fields. This article explored several physical systems, specifically

(i) Setting PID controllers utilising fractional computer principles;
(ii) Fragmentary P hexapod robot control;
(iii) Fragmentary dynamics of trajectory control of redundant manipulators;
(iv) Dissemination of heat;
(v) Synthesis circuit by means of evolutionary algorithms.

The benefits of this Mathematical Tool have been recognised for modelling and controlling such dynamic systems, and the results show the importance of fractional computation and drive new applications to emerge.

Although the indication of fractional calculus was started more than 300 years before, only just has serious efforts been devoted to its study. Still, normal calculus is much more acquainted, and more favoured, maybe because its applications are more seeming. However, it is the author's confidence that in addition to opening our minds to new branches of thought by satisfying the gaps of the normal calculus, fractional calculus has the possible of giving intriguing and useful applications in the future.

References:

1. R. Hilfer, Ed., *Applications of Fractional Calculus in Physics*, World Scientific Publishing, Singapore, 2000.

2. I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

3. M. F. Silva, J. A. T. Machado, and I. S. Jesus, "Modelling and simulation of walking robots with 3 dof legs," in *Proceedings of the 25th IASTED International Conference on Modelling, Identification and Control (MIC '06)*, pp. 271–276, Lanzarote, Spain, 2006.

4. M. F. Silva, J. A. T. Machado, and A. M. Lopes, "Position/force control of a walking robot," *Machine Intelligence and Robot Control*, vol. 5, pp. 33–44, 2003.

5. M. F. Silva and J. A. T. Machado, "Fractional order PDα joint control of legged robots," *Journal of Vibration and Control*, vol. 12, no. 12, pp. 1483–1501, 2006.

6. R. G. Roberts and A. A. Maciejewski, "Singularities, stable surfaces, and the repeatable behavior of kinematically redundant manipulators," *International Journal of Robotics Research*, vol. 13, no. 1, pp. 70–81, 1994.

7. M. da Graça Marcos, F. B. M. Duarte, and J. A. T. Machado, "Complex dynamics in the trajectory control of redundant manipulators," *Nonlinear Science and Complexity*, pp. 134–143, 2007.

8. K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, vol. 11 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1974.

9. I. S. Jesus, R. S. Barbosa, J. A. T. Machado, and J. B. Cunha, "Strategies for the control of heat diffusion systems based on fractional calculus," in *Proceedings of the IEEE International Conference on Computational Cybernetics (ICCC '06)*, Budapest, Hungary, 2006.